

# UNITS IN GROTHENDIECK-WITT RINGS AND $\mathbb{A}^1$ -SPHERICAL FIBRATIONS

MATTHIAS WENDT

**ABSTRACT.** In this note, we show that the units in Grothendieck-Witt rings extend to an unramified strictly  $\mathbb{A}^1$ -invariant sheaf of abelian groups on the category of smooth schemes. This implies that there is an  $\mathbb{A}^1$ -local classifying space of spherical fibrations.

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## 1. INTRODUCTION

In this note, we discuss units in Grothendieck-Witt rings and their relation to the theory of spherical fibrations. Via Morel's generalization of the Brouwer degree [Mor12, Corollary 1.24], the Grothendieck-Witt ring is the ring of continuous self-maps of an algebraic sphere up to  $\mathbb{A}^1$ -homotopy. A consequence of this is an identification of the group of homotopy self-equivalences of a sphere with the units in the Grothendieck-Witt ring. On the other hand, the general theory of localization of fibre sequences of simplicial sheaves developed in [Wen13] implies the existence of an  $\mathbb{A}^1$ -local classifying space of fibrations with fibre  $X$  provided the sheaf of homotopy self-equivalences is strongly  $\mathbb{A}^1$ -local. These two results are the starting point for the present work, in which we investigate strong  $\mathbb{A}^1$ -invariance of the sheaf of units in Grothendieck-Witt rings. The main result is the following:

**Theorem 1.** *Let  $k$  be an infinite perfect field of characteristic  $\neq 2$ . Associating to an extension field  $L/k$  the abelian group  $GW(L)^\times$  of units in the Grothendieck-Witt ring extends to a strictly  $\mathbb{A}^1$ -invariant sheaf of abelian groups on  $\mathrm{Sm}_k$ .*

There are several ingredients to the proof. First of all, the structure of the group of units in Grothendieck-Witt rings is well-known - it is a combination of square classes and units arising from torsion in the fundamental ideal. Moreover, for the additive groups of Grothendieck-Witt rings as well as the powers of the fundamental ideal, strict  $\mathbb{A}^1$ -invariance and the corresponding Gersten resolutions are

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already established [BGPW] resp. [Mor12, Chapters 4,5]. We describe a Gersten-type resolution of the units in the Grothendieck-Witt rings, based on the explicit computation of contractions.

It is noteworthy and strange that this Gersten resolution has underlying sets and set maps almost the Gersten resolution of the torsion in the fundamental ideal, but the abelian group structure of its terms is different from the usual addition in the Witt rings.

From the strict  $\mathbb{A}^1$ -invariance of the units in Grothendieck-Witt rings, we can establish the existence of an  $\mathbb{A}^1$ -local classifying space of spherical fibrations.

**Corollary 1.1.** *Let  $S^{2n,n}$  denote an  $\mathbb{A}^1$ -local fibrant model of the motivic sphere, and denote by  $B\mathrm{hAut}_\bullet(S^{2n,n})$  the corresponding classifying space of Nisnevich locally trivial torsors, as in [Wen11]. Then this space is in fact  $\mathbb{A}^1$ -local and classifies Nisnevich locally trivial spherical fibrations.*

The relation between strong  $\mathbb{A}^1$ -invariance of the units and the existence of a classifying space of spherical fibrations was already pointed out in [Wen13, Remark 8.3]. With this classifying space, we can discuss orientability and orientation characters for spherical fibrations. There is also an obvious analogue of the unstable  $J$ -homomorphism  $BGL_n \rightarrow B\mathrm{hAut}_\bullet(S^{2n,n})$ , whose homotopy fibre controls reduction of spherical fibrations to vector bundles. It turns out that non-trivial torsion in the Witt ring implies the existence of spherical fibrations which can not be reduced to vector bundles. This is a phenomenon which is not visible in either real or complex realization. In particular, we find exotic Poincaré dualities on the projective line  $\mathbb{P}^1$  over  $\mathbb{Q}$ , which are not induced from vector bundles.

*Structure of the paper:* We first recall some basic facts about the structure of Grothendieck-Witt rings in Section 2. In Section 3, we show that the units in the Grothendieck-Witt ring can be extended to an unramified sheaf of abelian groups. The main work is the computation of contractions in Section 4 and the description of the Gersten resolution of the units in Section 5. Finally, in Section 6, we discuss some consequences for orientation theory and spherical fibrations.

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## 2. RECOLLECTION ON GROTHENDIECK-WITT RINGS

We recall some basic facts about the structure of Grothendieck-Witt rings. The reference used will be [KK82]. The reader should be aware that there are more modern and much more high-tech approaches to Grothendieck-Witt groups available, cf. [Sch10]. In the present work, we will not need more than the structure of units and the existence of residue maps.

Throughout the paper we assume that the base field  $k$  has characteristic  $\neq 2$ .

**2.1. Basic definitions.** A bilinear space over  $k$  is a pair  $(V, \phi)$  of a  $k$ -vector space  $V$  and a symmetric bilinear form  $\phi : V \times V \rightarrow k$ . For example, the hyperbolic plane  $\mathbb{H}$  is the bilinear space  $(k^2, h)$  with the hyperbolic bilinear form  $h((x_1, x_2), (y_1, y_2)) = x_1y_2 + x_2y_1$ .

Denote by  $S(k)$  the set of isomorphism classes of bilinear spaces over  $k$ . Orthogonal sum of bilinear spaces defines an addition on  $S(k)$  with respect to which  $S(k)$  is a commutative monoid. There is also a multiplication given by tensor product of bilinear spaces:

$$(V_1, \phi_1) \otimes (V_2, \phi_2) = (V_1 \otimes V_2, \phi_1 \otimes \phi_2 : (e_1 \otimes e_2, f_1 \otimes f_2) \mapsto \phi_1(e_1, f_1)\phi_2(e_2, f_2)).$$

Two bilinear spaces  $V_1$  and  $V_2$  are called *stably isomorphic* if there exists a bilinear space  $W$  such that  $V_1 \oplus W \cong V_2 \oplus W$ .

**Definition 2.1.** *The Grothendieck-Witt ring  $GW(k)$  of the field  $k$  is the group completion of the commutative semiring of stable isomorphism classes of bilinear spaces over  $k$ . The Witt ring  $W(k)$  is the quotient of the Grothendieck-Witt ring by the ideal  $\mathbb{Z} \cdot [\mathbb{H}]$ .*

It can be shown, cf. [KK82], that the Grothendieck-Witt ring is the quotient of the group ring  $\mathbb{Z}[k^\times / (k^\times)^2]$  of the group of square classes modulo the relations

$$((1) + (a))((1) - (1 + a)), \quad a \in k^\times \setminus \{-1\}.$$

with  $(a)$  denoting the square class of  $a$  in the group ring. This description via the group ring of square classes is the basis for various structural theorems on Grothendieck-Witt rings, cf. [KK82].

Finally, note that in characteristic  $\neq 2$ , the Grothendieck-Witt ring can also be defined as the ring of stable isomorphism classes of quadratic spaces, i.e. vector spaces equipped with a quadratic form.

**2.2. Description of units.** Next, we recall from [KK82] the description of the units of the Witt ring of a field:

**Proposition 2.2.** *For every field  $F$  of characteristic  $\neq 2$ , there is a pushout square of abelian groups:*

$$\begin{array}{ccc} F^\times / (F^\times)^2 \cap (1 + I(F)_{\text{tor}}) & \longrightarrow & (1 + I(F)_{\text{tor}}) \\ \downarrow & & \downarrow \\ F^\times / (F^\times)^2 & \longrightarrow & W(F)^\times \end{array}$$

*Proof.* The bottom morphism  $F^\times / (F^\times)^2 \rightarrow W(F)^\times$  is the obvious one, mapping a square class  $a \in F^\times / (F^\times)^2$  to the class of the corresponding one-dimensional symmetric bilinear space in  $W(F)^\times$ .

The torsion of the fundamental ideal is a subgroup  $I(F)_{\text{tor}} \subseteq W(F)$ . As a consequence of [KK82, Propositions 2.12 and 2.15],  $I(F)_{\text{tor}} = \text{Nil}(W(F))$ . For  $x, y \in \text{Nil}(W(F)) = I(F)_{\text{tor}}$ , we have  $(1+x)(1+y) = 1+x+y+xy$  with  $x+y+xy \in \text{Nil}(W(F)) = I(F)_{\text{tor}}$ . Moreover, for  $x \in \text{Nil}(W(F))$  a nilpotent element,  $(1+x)$  is invertible - this provides the subgroup  $(1 + I(F)_{\text{tor}}) \hookrightarrow W(F)^\times$ .

By [KK82, Corollary 2.25] every unit of  $W(F)$  is of the form  $\pm \langle u \rangle (1+x)$  with  $u \in F^\times$  and  $x \in \text{Nil}(W(F))$  nilpotent. As in [KK82, Remark 2.26], the square classes in  $1 + \text{Nil}(W(F))$  are precisely the square classes  $\langle u \rangle$  with  $u$  a sum of squares.

The assertion then follows from these statements.  $\square$

We are actually interested in the units of the Grothendieck-Witt ring. We use the above quotient presentation  $GW(F) \rightarrow W(F)$ .

**Corollary 2.3.** *There is an exact sequence*

$$1 \rightarrow \{\pm 1\} \rightarrow GW(F)^\times \rightarrow W(F)^\times \rightarrow 1.$$

*Proof.* This follows from [KK82, Proposition 2.24] together with the fact that  $-1 \in GW(F)$  is not a square class. It becomes a square class in  $W(F)$  since  $-1 = \langle -1 \rangle$ .  $\square$

**Remark 2.4.** *Note that in general not all units of  $W(F)$  are represented by square classes. For example, if  $p$  is an odd prime, the cardinality of  $W(\mathbb{Q}_p)$  is 16, with  $W(\mathbb{Q}_p)^\times$  of size 8 but only 4 square classes. In this case, we have*

$$W(\mathbb{Q}_p)^\times / (\mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2) \cong (1 + I(\mathbb{Q}_p)_{\text{tor}}) / (\mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2 \cap (1 + I(\mathbb{Q}_p)_{\text{tor}})) \cong \mathbb{Z}/2.$$

*We describe a representative.*

Let  $a, b \in \mathbb{Q}_p$  with  $(a, b)_p = -1$  and consider the norm form of the quaternion algebra for  $(a, b)$ :

$$f(X, Y, Z, T) = Z^2 - aX^2 - bY^2 + abT^2.$$

This is an element of  $I(\mathbb{Q}_p)_{\text{tor}}$  and has invariants  $d(f) = 1$  and  $\epsilon(f) = -(-1, -1)$ . On the other hand, any 5-dimensional form represented by a square class is of the form  $g \sim [a] \oplus \mathbb{H}^2$ , and therefore has invariants  $d(g) = a$  and  $\epsilon(g) = (-1, -1)$ . We see that  $1 \oplus f$  - which has the same invariants as  $f$  - is a unit of  $W(\mathbb{Q}_p)$  not represented by a square class.

Of course, similar examples can be constructed over  $\mathbb{Q}$  using the Hasse-Minkowski local-global principle. This way, we see that  $W(\mathbb{Q})^\times / (\mathbb{Q}^\times / (\mathbb{Q}^\times)^2)$  is infinite.

Conversely, note that  $I(\mathbb{R})_{\text{tor}} = 0$ , therefore of the units in  $W(\mathbb{R}) = \mathbb{Z}$ , one is of the form  $1 + x$  with  $x$  nilpotent, but  $-1$  is not.

### 3. THE UNITS AS UNRAMIFIED SHEAF

In this section, we will show that the units in Grothendieck-Witt rings extend to an unramified sheaf of abelian groups on the category  $\text{Sm}_k$  of smooth schemes over the field  $k$ , in the sense of [Mor12, Definition 2.1]. For that, we use the correspondence between unramified sheaves and unramified  $\mathcal{F}_k$ -data from [Mor12, Section 2]. The data used for the units (i.e. residue and specialization morphisms) are the same as those for the Grothendieck-Witt rings, which are in fact compatible with the ring structure.

In this section, we assume the base field to be infinite perfect, and we denote by  $\mathcal{F}_k$  the category of field extensions  $L/k$  such that  $L$  has finite transcendence degree over  $k$ . For a discrete valuation  $v$  on the field  $L$ , we denote by  $\kappa(v)$  the corresponding residue field.

**3.1. Recollection on unramified sheaves.** We just recall the main points of [Mor12, Section 2.1]. The central definitions in [Mor12, Section 2.1] are the definition of *unramified presheaves of sets* in Definition 2.1, and the definition of *unramified  $\mathcal{F}_k$ -data* in the combined Definition 2.6 and Definition 2.9. The main result of the section is Theorem 2.11 which states that there is an equivalence of categories between the unramified  $\mathcal{F}_k$ -data and unramified sheaves of sets on  $\text{Sm}_k$ . The functor from unramified sheaves on  $\text{Sm}_k$  to  $\mathcal{F}_k$ -data is given by evaluation - all finitely generated field extensions of  $k$  are function fields of smooth schemes over  $k$ . The functor in the other direction, from  $\mathcal{F}_k$ -data to unramified sheaves on  $\text{Sm}_k$  is given by taking the unramified elements: for irreducible  $X$  we take

$$X \mapsto \mathcal{S}(X) = \bigcap_{x \in X^{(1)}} \mathcal{S}(\mathcal{O}_x) \subseteq \mathcal{S}(k(X))$$

and for general  $X \in \text{Sm}_k$ , we set  $\mathcal{S}(X) = \prod_{i \in X^{(0)}} \mathcal{S}(X_i)$ .

**3.2. Residue and specialization morphisms.** In this section, we recall the residue morphisms on Grothendieck-Witt rings. We will mostly follow [Mil69] and [Mor12]. The residue morphisms provide us with the data (D1)-(D3) for unramified  $\mathcal{F}_k$ -data. The first such datum is obvious - (D1) just requires that we have induced morphisms for field extensions. The obvious functorial homomorphisms for Grothendieck-Witt rings are ring homomorphisms, which provides data (D1) for the units of Grothendieck-Witt rings. We note that  $GW^\times$  is a continuous functor: the Grothendieck-Witt rings are continuous functors, therefore also  $GW^\times$  commutes with filtered colimits.

For the datum (D2), we need to specify the unramified elements. Recall from [Mor12, Theorem 3.15] that there are residue homomorphisms  $\partial_v^\pi : K_\bullet^{MW}(F) \rightarrow K_{\bullet-1}^{MW}(\kappa(v))$  for any discrete valuation  $v$  on  $F$  with valuation ring  $\mathcal{O}_v$ , residue field

$\kappa(v)$  and choice of uniformizer  $\pi$ . By [Mor12, Lemma 3.19], the kernel of such a residue homomorphism does not depend on the choice of uniformizer  $\pi$ , and is denoted by  $\underline{\mathbf{K}}_n^{MW}(\mathcal{O}_v)$ . Finally, [Mor12, Theorem 3.22] states that this  $\underline{\mathbf{K}}_\bullet^{MW}(\mathcal{O}_v)$  is the subring generated by  $\eta$  and the symbols  $[u] \in K_1^{MW}(F)$  with  $u \in \mathcal{O}_v^\times$ . Therefore, we can set

$$\mathbf{GW}^\times(\mathcal{O}_v) = \underline{\mathbf{K}}_0^{MW}(\mathcal{O}_v)^\times.$$

In particular, the unramified elements form an abelian subgroup of  $\mathbf{GW}(F)^\times$ .

Finally, we need the datum (D3), i.e. for any  $F \in \mathcal{F}_k$  and any discrete valuation  $v$  on  $F$  a specialization morphism  $s_v : \mathbf{GW}^\times(\mathcal{O}_v) \rightarrow \mathbf{GW}^\times(\kappa(v))$ . For this, we have to unwind the proof of [Mor12, Lemma 2.36], because the fact that Milnor-Witt K-theory sheaves are unramified is deduced from other axioms which only make sense in the  $\mathbb{Z}$ -graded situation which we do not have here. The datum (D3) on  $\underline{\mathbf{K}}_0^{MW}$  is given by

$$s_v : \underline{\mathbf{K}}_0^{MW}(\mathcal{O}_v) \rightarrow K_0^{MW}(\kappa(v)) : \alpha \mapsto \partial_v^\pi([\pi]\alpha)$$

and is in fact independent of the choice of a  $\pi$ . We compute the value of  $s_v^\pi$  on generators  $1 + \eta[u]$  with  $u \in \mathcal{O}_v^\times$ . From [Mor12, Theorem 3.15], we find

$$s_v^\pi(\langle u \rangle) = \partial_v^\pi([\pi](1 + \eta[u])) = 1 + \eta[\bar{u}] = \langle \bar{u} \rangle.$$

From [Mor12, Lemma 3.16], this is in fact a morphism of rings on  $\underline{\mathbf{K}}_0^{MW}(\mathcal{O}_v)$ . In particular, the reduction morphism  $\mathcal{O}_v^\times \rightarrow \kappa(v)^\times$  induces a morphism of units

$$s_v : \underline{\mathbf{K}}_0^{MW}(\mathcal{O}_v)^\times \rightarrow K_0^{MW}(\kappa(v))^\times.$$

This is the relevant datum (D3).

**Remark 3.1.** As in [Mor12, Theorem 3.22],  $\underline{\mathbf{K}}_0^{MW}(\mathcal{O}_v)$  is the subring of  $K_0^{MW}(F)$  generated by the symbols  $\langle u \rangle = 1 + \eta[u]$  with  $u \in \mathcal{O}_v^\times$ . In particular, the morphism

$$\mathbb{Z}[\mathcal{O}_v^\times / (\mathcal{O}_v^\times)^2] \rightarrow K_0^{MW}(F) : u \mapsto \langle u \rangle$$

factors through  $\underline{\mathbf{K}}_0^{MW}(\mathcal{O}_v)$ . It is not clear to me if the relations usually imposed in the definition of the Grothendieck-Witt ring generate the kernel of the above morphism.

**Proposition 3.2.** Let  $F \in \mathcal{F}_k$ , and let  $v$  be a discrete valuation on  $F$ . We have

$$\mathbf{GW}(F)^\times \cap \mathbf{GW}(\mathcal{O}_v) = \mathbf{GW}^\times(\mathcal{O}_v).$$

*Proof.* Only the  $\subseteq$ -direction needs a proof. We use the square of Proposition 2.2. For units coming from  $F^\times / (F^\times)^2$ , the inclusion is true: if such a unit is unramified, then it is the square class of a unit in  $\mathcal{O}_v$ , hence invertible in  $\mathbf{GW}(\mathcal{O}_v)$ .

Now let  $1 + a \in \mathbf{GW}(F)^\times$ , i.e.  $a \in I(F)_{\text{tor}}$ . Assume  $1 + a \in \mathbf{GW}(\mathcal{O}_v)$ , hence  $a \in \mathbf{I}_{\text{tor}}(\mathcal{O}_v)$ . Its inverse in  $\mathbf{GW}(F)^\times$  is given by  $1 + \sum_{i=1}^{n-1} (-a)^i$ , where  $n$  is the smallest positive number such that  $a^n = 0$ . But then  $\sum_{i=1}^{n-1} (-a)^i \in \mathbf{I}_{\text{tor}}(\mathcal{O}_v)$ , hence  $1 + a \in \mathbf{GW}^\times(\mathcal{O}_v)$ .  $\square$

**3.3. Verification of the axioms.** Our next goal is the verification of the axioms (A1)-(A4) for unramified  $\mathcal{F}_k$ -data from [Mor12, Chapter 2].

**Lemma 3.3** (Axiom (A1)). Let  $\iota : E \hookrightarrow F$  be a separable extension in  $\mathcal{F}_k$ , and let  $v$  be a discrete valuation on  $F$  which restricts to a discrete valuation  $w$  on  $E$  with ramification index 1. Then  $\mathbf{GW}(\iota)^\times$  maps  $\mathbf{GW}^\times(\mathcal{O}_w)$  into  $\mathbf{GW}^\times(\mathcal{O}_v)$ . If furthermore  $\iota$  induces an isomorphism  $\bar{\iota} : \kappa(w) \rightarrow \kappa(v)$  of residue fields, then the following square is cartesian:

$$\begin{array}{ccc}
\mathbf{GW}^\times(\mathcal{O}_w) & \longrightarrow & \mathbf{GW}^\times(\mathcal{O}_v) \\
\downarrow & & \downarrow \\
GW(E)^\times & \longrightarrow & GW(F)^\times
\end{array}$$

*Proof.* As  $\mathbf{GW}(\mathcal{O}_w) = \underline{\mathbf{K}}_0^{MW}(\mathcal{O}_w)$  is the subring generated by symbols  $\langle u \rangle$  with  $u \in \mathcal{O}_w$  and the ring homomorphism  $GW(E) \rightarrow GW(F)$  is induced by  $\langle x \rangle \mapsto \langle \iota(x) \rangle$  for  $x \in E^\times$ , we find that  $GW(E)^\times \rightarrow GW(F)^\times$  restricts to a homomorphism  $\mathbf{GW}^\times(\mathcal{O}_w) \rightarrow \mathbf{GW}^\times(\mathcal{O}_v)$ .

The second assertion states that  $\mathbf{GW}^\times(\mathcal{O}_w)$  is the preimage of  $\mathbf{GW}^\times(\mathcal{O}_v)$  under the homomorphism  $GW(E)^\times \rightarrow GW(F)^\times$ . Axiom (A1) for  $GW$  states that  $\mathbf{GW}(\mathcal{O}_w)$  is the preimage of  $\mathbf{GW}(\mathcal{O}_v)$  under the ring homomorphism  $GW(E) \rightarrow GW(F)$ . Then we apply Proposition 3.2. Alternatively, given an element  $x \in GW(E)^\times$  which under the homomorphism lands in  $\mathbf{GW}^\times(\mathcal{O}_v)$ . This means that both  $x$  and  $x^{-1}$  are unramified in  $GW(F)$ . By Axiom (A1) for  $GW$ , both  $x$  and  $x^{-1}$  lie in  $\mathbf{GW}(\mathcal{O}_w)$ , hence  $x \in \mathbf{GW}^\times(\mathcal{O}_w)$ .  $\square$

**Lemma 3.4** (Axiom (A2)). *For an irreducible smooth scheme  $X$  over  $k$  with function field  $F$ , every element  $x \in GW(F)^\times$  lies in all but a finite number of  $\mathbf{GW}^\times(\mathcal{O}_v)$ , with  $v \in X^{(1)}$  running through the (valuations for) codimension 1 points of  $X$ .*

*Proof.* Any element in  $K_0^{MW}(F)$  lies in all but a finite number of  $\underline{\mathbf{K}}_0^{MW}(\mathcal{O}_v)$ . For  $x \in GW(F)^\times$ , both  $x$  and  $x^{-1}$  lie in all but a finite number of  $\mathbf{GW}^\times(\mathcal{O}_v)$ , proving the claim.  $\square$

**Lemma 3.5** (Axiom (A3)). *Let  $\iota : E \hookrightarrow F$  be a separable extension in  $\mathcal{F}_k$ , and let  $v$  be a discrete valuation on  $F$ .*

- (i) *If  $v$  restricts to a discrete valuation  $w$  on  $E$  with ramification index 1, then the following diagram is commutative:*

$$\begin{array}{ccc}
\mathbf{GW}^\times(\mathcal{O}_w) & \longrightarrow & \mathbf{GW}^\times(\mathcal{O}_v) \\
s_w \downarrow & & \downarrow s_v \\
GW(\kappa(w))^\times & \longrightarrow & GW(\kappa(v))^\times
\end{array}$$

- (ii) *If  $v$  restricts to 0 on  $E$ , then the image of  $GW(E)^\times$  is contained in  $\mathbf{GW}^\times(\mathcal{O}_v)$  and - denoting  $j : E \hookrightarrow \kappa(v)$  - we have a commutative triangle:*

$$\begin{array}{ccc}
GW(E)^\times & \longrightarrow & \mathbf{GW}^\times(\mathcal{O}_v) \\
& \searrow GW(j) & \swarrow s_v \\
& & GW(\kappa(v))^\times
\end{array}$$

*Proof.* Part (i) again can be verified using the whole Grothendieck-Witt ring. The ring  $\mathbf{GW}(\mathcal{O}_w)$  is generated by elements of the form  $\langle u \rangle$  with  $u \in \mathcal{O}_w$ . Commutativity of the corresponding diagram for  $GW$  instead of  $GW^\times$  reduces to the obvious  $\iota(\overline{\iota(u)}) = \overline{\iota(u)}$ . Since all morphisms in the diagram for  $GW$  respect the ring structure, we obtain a commutative diagram of units, as required.

For part (ii), we note again that the group  $GW(E)$  is generated by  $\langle u \rangle$  with  $u \in E^\times$ . Since  $v(u) = 0$ , i.e.  $\iota(u) \in \mathcal{O}_v^\times$ , we have  $\langle \iota(u) \rangle \in \underline{\mathbf{K}}_0^{MW}(\mathcal{O}_v)$ . In particular the image  $GW(E)^\times$  in  $GW(F)^\times$  is contained in  $\underline{\mathbf{K}}_0^{MW}(\mathcal{O}_v)^\times$ . The homomorphism denoted by  $s_v \circ \mathcal{S}(i)$  in [Mor12] is therefore  $\langle u \rangle \mapsto \langle \iota(u) \rangle$ . The homomorphism denoted by  $j : E \rightarrow \kappa(v)$  equals  $u \mapsto \overline{\iota(u)}$ . Again, everything is compatible with the ring structures, hence we have verified (A3ii).  $\square$

**Lemma 3.6** (Axiom (A4)). *(i) Let  $X$  be an essentially smooth local scheme of dimension 2, let  $z$  be the closed point of  $X$  and let  $y_0$  be a codimension 1 point with essentially smooth closure. Then the specialization  $s_{y_0} : \mathbf{GW}^\times(\mathcal{O}_{y_0}) \rightarrow GW(\kappa(y_0))^\times$  maps  $\bigcap_{y \in X^{(1)}} \mathbf{GW}^\times(\mathcal{O}_y)$  into  $\mathbf{GW}^\times(\mathcal{O}_{\overline{y_0}, z}) \subseteq GW(\kappa(y_0))^\times$ .*  
*(ii) The composition*

$$\bigcap_{y \in X^{(1)}} \mathbf{GW}^\times(\mathcal{O}_y) \rightarrow \mathbf{GW}^\times(\mathcal{O}_{\overline{y_0}, z}) \rightarrow \mathbf{GW}^\times(\kappa(z))$$

*does not depend on the choice of a codimension 1 point  $y_0$  with essentially smooth closure.*

*Proof.* Again, we know Axiom (A4) for the unramified sheaf  $\mathbf{GW}$  of Grothendieck-Witt rings.

In (i), the specialization morphism  $\mathbf{GW}(\mathcal{O}_{y_0}) \rightarrow GW(\kappa(y_0))$  is a ring homomorphism, cf. [Mor12, Lemma 3.16], and the specialization morphisms for units  $s_{y_0} : \mathbf{GW}^\times(\mathcal{O}_{y_0}) \rightarrow GW(\kappa(y_0))^\times$  is induced from the corresponding specialization morphism for  $GW$ . By Axiom (A4) for  $\mathbf{GW}$ ,  $\bigcap_{y \in X^{(1)}} \mathbf{GW}(\mathcal{O}_y)$  lands in  $\mathbf{GW}(\mathcal{O}_{\overline{y_0}, z}) \subseteq GW(\kappa(y_0))$ . Therefore,  $\mathbf{GW}^\times(\mathcal{O}_{y_0})$  lands in  $\mathbf{GW}(\mathcal{O}_{\overline{y_0}, z}) \cap GW(\kappa(y_0))^\times$ . The conclusion follows from Proposition 3.2.

The composition in (ii) is a composition of specialization morphisms. As mentioned before, these are the restrictions of the specialization morphisms from  $\mathbf{GW}$  to the group of units. Axiom (A4ii) for  $\mathbf{GW}$  states that the composition

$$\bigcap_{y \in X^{(1)}} \mathbf{GW}(\mathcal{O}_y) \rightarrow \mathbf{GW}(\mathcal{O}_{\overline{y_0}, z}) \rightarrow \mathbf{GW}(\kappa(z))$$

is independent of the choice of  $y_0$ . Therefore, so is the restriction of this morphism to the groups of units.  $\square$

Having verified the axioms, the following is now a consequence of [Mor12, Theorem 2.11].

**Proposition 3.7.** *Let  $k$  be an infinite perfect field of characteristic  $\neq 2$ . The assignment*

$$GW^\times : \mathcal{F}_k \rightarrow \mathbf{Set} : L \mapsto GW(L)^\times$$

*together with the data (D1)-(D3) at the beginning of the section satisfy the axioms for an unramified  $\mathcal{F}_k$ -datum. In particular,  $GW^\times$  extends to an unramified sheaf of abelian groups on  $\mathbf{Sm}_k$ .*

**Remark 3.8.** *Note that what we have proved above is: there is a reasonable notion of unramified sheaf of rings in which all the data (D1)-(D3) are compatible with the ring structures, and an unramified sheaf of rings has an unramified sheaf of units. In lack of examples other than the one above, we chose not to axiomatize this.*

#### 4. CONTRACTIONS

In this section, we want to compute the contractions of the unramified sheaf  $\mathbf{GW}^\times$ . Recall from [Mor12], that for a presheaf of groups  $\mathcal{G}$  on  $\mathbf{Sm}_k$ , one can define its contraction

$$\mathcal{G}_{-1} : \mathbf{Sm}_k^{\text{op}} \rightarrow \mathcal{G}rp : X \mapsto \ker(\text{ev}_1 : \mathcal{G}(\mathbb{G}_m \times X) \rightarrow \mathcal{G}(X)),$$

where the morphism  $\text{ev}_1$  is induced from the inclusion  $1 : \text{Spec}(k) \hookrightarrow \mathbb{G}_m$ .

In the course of computing contractions, we will need a modified version of Witt groups. We introduce some notation:

**Definition 4.1.** Let  $k$  be a field. We denote by  $W(k)_{\text{tor}}$  the torsion subgroup of the Witt ring. For  $n \in \mathbb{N}$ , we denote by  $W(k)_{\text{tor}}^{(n)}$  the set of torsion elements in  $W(k)$  with the following modified addition

$$a \boxplus_n b = a + b + (-2)^n \cdot a \cdot b,$$

where  $+$  and  $\cdot$  on the right are the usual addition and multiplication in the Witt ring.

**Lemma 4.2.** The addition  $\boxplus_n$  defined above turns  $W(k)_{\text{tor}}^{(n)}$  into an abelian group.

*Proof.* The product of torsion elements is again a torsion element, hence the modified addition is in fact an operation on  $W(k)_{\text{tor}}$ . Associativity is

$$\begin{aligned} (a \boxplus_n b) \boxplus_n c &= a + b + c + (-2)^n (ab + ac + bc + (-2)^n abc) \\ &= a \boxplus_n (b \boxplus_n c). \end{aligned}$$

The neutral element is 0, since  $0 \boxplus_n a = 0 + a + (-2)^n \cdot 0 \cdot a$ . Commutativity is clear from the defining formula and commutativity of the Witt ring multiplication. Finally, the  $\boxplus_n$ -inverse of  $a$  is given by

$$- \sum_{i=1}^j (-1)^{(i-1)(n-1)} 2^{n(i-1)} a^i.$$

This is the usual formula for inverting a nilpotent element, the bound  $j$  for the summation is the smallest natural number such that  $(-1)^{(j-1)(n-1)} 2^{nj} a^{j+1} = 0$ .  $\square$

In the following, we use the notation  $H_v^1$  from [Mor12, Section 3].

**Lemma 4.3.** Let  $F$  be a field in  $\mathcal{F}_k$ , let  $v$  be a discrete valuation on  $F$  and let  $\pi$  be a uniformizing element for  $v$ . Then there exist isomorphisms

$$\begin{aligned} (i) \quad H_v^1(\mathcal{O}_v, \mathbb{G}_m/2) &= F^\times / ((F^\times)^2 \cdot \mathcal{O}_v^\times) \xrightarrow{\cong} \mathbb{Z}/2, \text{ and} \\ (ii) \quad H_v^1(\mathcal{O}_v, 1 + \mathbf{I}_{\text{tor}}) &= (1 + I(F)_{\text{tor}}) / (1 + \mathbf{I}_{\text{tor}}(\mathcal{O}_v)) \xrightarrow{\cong} W(\kappa(v))_{\text{tor}}^{(1)}. \end{aligned}$$

*Proof.* The isomorphism (i) is clear: every element of  $x \in F^\times$  can be written as  $x = \pi^n \cdot u$  with  $u \in \mathcal{O}_v^\times$ , and the isomorphism is given by mapping  $x = \pi^n u \mapsto n \pmod{2}$ .

For (ii), we need a little more work. Recall (e.g. from [Mil69, Proof of Corollary 5.2]) the definition of the residue homomorphism

$$\partial : I(F) \xrightarrow{\partial} W(\kappa(v)) : \langle \pi^n u \rangle \mapsto \begin{cases} \langle \bar{u} \rangle & 2 \nmid n \\ 0 & \text{otherwise} \end{cases}$$

This is evidently an epimorphism, and its kernel is  $\mathbf{I}(\mathcal{O}_v)$ . The residue morphism restricts to a map

$$\partial : 1 + I(F)_{\text{tor}} \rightarrow W(\kappa(v))_{\text{tor}}^{(1)} : 1 + a \mapsto \partial(a),$$

where the group structure on the source is multiplication in the Witt ring, the group structure on the target is the modified addition of Definition 4.1. We need to establish that we have a homomorphism. First note that  $GW(F)$  is generated by  $\langle x \rangle$  with  $x \in F^\times$ . Any such  $x$  can be written as  $x = \pi^n u$  with  $u \in \mathcal{O}_v^\times$ . Since  $\langle \pi^2 \rangle = 1$ , any element of  $1 + I(F)_{\text{tor}}$  can be written as follows:

$$\begin{aligned} 1 + a + \langle \pi \rangle b &= 1 + a + b + (\langle \pi \rangle - 1)b \\ &\equiv 1 + (\langle \pi \rangle - 1)b(1 + a + b)^{-1} \pmod{\mathbf{GW}^\times(\mathcal{O}_v)}. \end{aligned}$$

Note that we have used here that if  $a + \langle \pi \rangle b$  is torsion, then (via  $\partial$ ) also  $b$  and hence  $a$  are torsion, so  $1 + a + b$  is in fact invertible. The above decomposition is unique



up to multiplication with elements from  $\mathbf{GW}^\times(\mathcal{O}_v)$ . Now let  $(1 + (\langle \pi \rangle - 1)a)$  and  $(1 + (\langle \pi \rangle - 1)b)$  be such elements in  $1 + I(F)_{\text{tor}}$ . We find

$$\begin{aligned} (1 + (\langle \pi \rangle - 1)a) \cdot (1 + (\langle \pi \rangle - 1)b) &= 1 + (\langle \pi \rangle - 1)(a + b) + (\langle \pi \rangle - 1)^2 ab \\ &= 1 + (\langle \pi \rangle - 1)(a + b - 2ab), \end{aligned}$$

where we use  $(\langle \pi \rangle - 1)^2 = 2 - 2\langle \pi \rangle$ . Applying  $\partial$  to the above equation, we have on the left side the reduction of the product, and on the right side, we have  $\boxplus_1$ -sum  $a + b - 2ab$ . Hence,  $\partial$  is a homomorphism.

For injectivity, we note that the kernel of  $\partial$  is identified with  $1 + \mathbf{I}_{\text{tor}}(\mathcal{O}_v)$ , since the set  $\mathbf{I}_{\text{tor}}(\mathcal{O}_v)$  of unramified elements of  $I(F)_{\text{tor}}$  is exactly the kernel of  $\partial : I(F)_{\text{tor}} \rightarrow W(\kappa(v))_{\text{tor}}$ .

Finally, we need surjectivity of  $\partial$ . Let  $\bar{\alpha} \in W(\kappa(v))_{\text{tor}}$  be an element. The ring  $\mathcal{O}_v$  is smooth and essentially of finite type over a field  $L \subseteq \mathcal{O}_v$ . By [Lin81, Proposition 2], there exists a subring  $A \hookrightarrow \mathcal{O}_v$  such that  $A = L[X]_{\mathfrak{m}}$  with  $\mathfrak{m} \subseteq L[X]$  a maximal ideal. Moreover, the inclusion  $A \subseteq \mathcal{O}_v$  has the following properties: in the corresponding field extension  $\text{frac}(A) \hookrightarrow F$ , the valuation  $v$  on  $F$  restricts to the valuation of  $\text{frac}(A)$  having  $A$  as valuation ring, the ramification index is 1 and the inclusion  $A \hookrightarrow \mathcal{O}_v$  induces an isomorphism of residue fields. Now we use Milnor's split exact sequence [Mil69, Theorem 5.3]

$$0 \rightarrow I(L) \rightarrow I(L(T)) \rightarrow \bigoplus_{\pi} W(L[T]/(\pi)) \rightarrow 0.$$

Let  $\pi \in L[T]$  be a generator of the maximal ideal  $\mathfrak{m}$  above. The element  $\bar{\alpha}$  can be viewed as an element of  $W(L[T]/(\pi))_{\text{tor}}$ , since the inclusion  $A \hookrightarrow \mathcal{O}_v$  induces an isomorphism of residue fields. Because Milnor's sequence is split exact, there exists an element  $\alpha \in I(L(T))_{\text{tor}}$  mapping to  $\bar{\alpha} \in W(L[T]/(\pi))_{\text{tor}}$  under the corresponding residue homomorphism. The image of  $1 + \alpha$  under the canonical homomorphism  $1 + I(\text{frac}(A))_{\text{tor}} \rightarrow 1 + I(K)_{\text{tor}}$  provides a  $\partial$ -preimage of  $\bar{\alpha}$ .  $\square$

Note that the first isomorphism in the statement is independent of the choice of a  $\pi$ , the second is not.

**Proposition 4.4.** *Let  $F$  be a field in  $\mathcal{F}_k$ , let  $v$  be a discrete valuation on  $F$  and let  $\pi$  be a uniformizing element for  $v$ . Then there exists an isomorphism*

$$H_v^1(\mathcal{O}_v, \mathbf{GW}^\times) = GW(F)^\times / \mathbf{GW}^\times(\mathcal{O}_v) \xrightarrow{\cong} \mathcal{A} \oplus W(\kappa(v))_{\text{tor}}^{(1)},$$

where  $\mathcal{A} = 0$  if  $\pi$  is a sum of squares in  $F$ , and  $\mathcal{A} \cong \mathbb{Z}/2$  otherwise.

*Proof.* We first prove the result for  $H_v^1(\mathcal{O}_v, \mathbf{W}^\times) = W(F)^\times / \mathbf{W}^\times(\mathcal{O}_v)$  instead. In that case, the result will follow from the pushout square of Proposition 2.2 and the previous lemma. First, we claim that the intersection of the pushout square with  $\mathbf{W}^\times(\mathcal{O}_v)$  results in the corresponding pushout square of the unramified subgroups:

$$\begin{array}{ccc} \mathcal{O}_v^\times / (\mathcal{O}_v^\times)^2 \cap (1 + \mathbf{I}_{\text{tor}}(\mathcal{O}_v)) & \longrightarrow & (1 + \mathbf{I}_{\text{tor}}(\mathcal{O}_v)) \\ \downarrow & & \downarrow \\ \mathcal{O}_v^\times / (\mathcal{O}_v^\times)^2 & \longrightarrow & \mathbf{W}^\times(\mathcal{O}_v) \end{array}$$

An inspection of the proof of [KK82, Proposition 2.24] shows that the unramified units are precisely those that can be written as product of an unramified square class and an unramified element from  $(1 + \mathbf{I}_{\text{tor}})$ : if a unit is unramified, we can write it as sum  $\sum \langle u_i \rangle$  with  $u_i \in \mathcal{O}_v^\times$ . In loc.cit., the element can be decomposed multiplicatively as the product of the square class  $(-1)^n \prod \langle u_i \rangle$  and an element of the form  $1 + a$  with  $a$  nilpotent. In particular, both factors have to be unramified.

Therefore, intersecting the pushout square with  $\mathbf{W}^\times(\mathcal{O}_v)$  indeed produces exactly the unramified subgroups.

It follows, that the quotient of the squares is also a pushout square:

$$\begin{array}{ccc} \mathcal{S} & \longrightarrow & H_v^1(\mathcal{O}_v, 1 + \mathbf{I}_{\text{tor}}) \cong W(\kappa(v))_{\text{tor}}^{(1)} \\ \downarrow & & \downarrow \\ H_v^1(\mathcal{O}_v, \mathbb{G}_m/2) \cong \mathbb{Z}/2 & \longrightarrow & H_v^1(\mathcal{O}_v, \mathbf{W}^\times) \end{array}$$

The identifications in the lower left and upper right corner follow from Lemma 4.3. Similarly, the  $\mathcal{S}$  in the upper left corner is  $H_v^1(\mathcal{O}_v, \mathbb{G}_m/2 \cap (1 + \mathbf{I}_{\text{tor}}))$ . As can be seen from the first isomorphism of Lemma 4.3, this is  $\mathbb{Z}/2$  or 0 if  $\pi \in (1 + I(F)_{\text{tor}})$  or not, respectively. But the intersection  $F^\times / (F^\times)^2 \cap (1 + I(F)_{\text{tor}})$  consists precisely of the sums of squares. Hence the statement for  $\mathbf{W}^\times$ .

The statement for  $\mathbf{GW}^\times$  follows from this together with Corollary 2.3, since  $-1 \in GW(F)$  is unramified, hence does not contribute to the quotient.  $\square$

**Remark 4.5.** *The above implies in particular that we have computed the first contraction  $\mathbf{W}^\times_{-1}$ . In that language, the above result reads*

$$\mathbf{W}^\times_{-1}(L) \cong \mathcal{A} \oplus W(L)_{\text{tor}}^{(1)}.$$

**Proposition 4.6.** *Let  $F$  be a field in  $\mathcal{F}_k$ . Then we have the following short exact sequence*

$$1 \rightarrow GW(F)^\times \rightarrow GW(F(T))^\times \xrightarrow{\sum \partial_{(\pi)}^\pi} \bigoplus_{\pi \in F[T]} \left( \mathcal{A}_\pi \oplus W(F[T]/(\pi))_{\text{tor}}^{(1)} \right) \rightarrow 0,$$

where the direct sum is taken over all irreducible monic polynomials.

*Proof.* As in the proof of Proposition 4.4, it suffices to consider  $W(F)^\times$  instead of  $GW(F)^\times$ . Recall Milnor's split exact sequence [Mil69, Theorem 5.3]

$$0 \rightarrow W(F) \rightarrow W(F(T)) \rightarrow \bigoplus_{\pi} W(F[T]/(\pi)) \rightarrow 0.$$

This immediately settles injectivity  $W(F)^\times \hookrightarrow W(F(T))^\times$ .

For surjectivity of  $\sum \partial_{(\pi)}^\pi$ , let  $\bar{\alpha} \in \bigoplus_{\pi \in F[T]} W(F[T]/(\pi))$  be torsion. Using the splitting of Milnor's sequence, there exists  $\alpha \in W(F(T))_{\text{tor}}$  lifting  $\bar{\alpha}$ . Setting  $1 + \alpha \in W(F(T))^\times$ , we have

$$\left( \sum \partial_{(\pi)}^\pi \right) (1 + \alpha) = \bar{\alpha}.$$

If  $\pi$  is such that  $\mathcal{A}_\pi$  is  $\mathbb{Z}/2$ , then  $\langle \pi \rangle$  provides a lift of  $1 \in \mathcal{A}_\pi$ .

Finally, exactness in the middle. Let  $\langle u \rangle \in F(T)^\times / (F(T)^\times)^2$  be a square class unit with  $(\sum \partial_{(\pi)}^\pi)(\langle u \rangle) = 0$ . Decomposing  $u$  into primes, it suffices to consider the two cases  $u = \pi$  irreducible monic and  $u \in F^\times$  constant. In the second case,  $u$  is obviously in the image and we are done. Let  $1 + a \in (1 + I(F(T))_{\text{tor}})$  with  $(\sum \partial_{(\pi)}^\pi)(1 + a) = 0$ . By Proposition 4.4, this means that  $a$  maps to 0 under the residue morphism  $W(F(T)) \rightarrow \bigoplus_{\pi} W(F[T]/(\pi))$  in Milnor's sequence. Exactness of Milnor's sequence implies that  $a$  is in the image of  $W(F)_{\text{tor}}$ . Hence the unit  $1 + a$  lies in the image of  $W(F)^\times$ .  $\square$

We record a similar but much easier computation for the torsion in the Witt groups. This will lead to an identification of the higher contractions of the units.

**Proposition 4.7.** *Let  $F$  be a field in  $\mathcal{F}_k$ , let  $v$  be a discrete valuation on  $F$  and let  $\pi$  be a uniformizing element for  $v$ . Then there exists an isomorphism*

$$H_v^1(\mathcal{O}_v, \mathbf{W}_{\text{tor}}^{(n)}) = W(F)_{\text{tor}}^{(n)} / \mathbf{W}_{\text{tor}}^{(n)}(\mathcal{O}_v) \xrightarrow{\cong} W(\kappa(v))_{\text{tor}}^{(n+1)}.$$

Moreover, there is a short exact sequence

$$1 \rightarrow W(F)_{\text{tor}}^{(n)} \rightarrow W(F(T))_{\text{tor}}^{(n)} \xrightarrow{\sum \partial_{(\pi)}} \bigoplus_{\pi \in F[T]} W(F[T]/(\pi))_{\text{tor}}^{(n+1)} \rightarrow 0,$$

where the direct sum is taken over all irreducible monic polynomials.

*Proof.* The second part without the modifying  $(n)$ -s is a direct consequence of Milnor's split exact sequence [Mil69, Theorem 5.3], restricted to the torsion.

For the first part, note that the morphism  $W(F)_{\text{tor}} \rightarrow W(\kappa(v))_{\text{tor}}$  is simply the residue restricted to torsion, and it is split by  $a \mapsto (\langle \pi \rangle - 1)a$ . As a set map, it is the same residue morphism as for the Witt groups with the usual addition. Therefore, as a set map, injectivity of  $\theta_\pi$  is the Gersten conjecture for Witt groups in a very simple case, surjectivity follows as in Lemma 4.3.

We need to check the homomorphism property. By the form of the splitting mentioned above, we need to compute the product

$$\begin{aligned} ((\langle \pi \rangle - 1)a) \boxplus_n ((\langle \pi \rangle - 1)b) &= (\langle \pi \rangle - 1)(a + b) + (-2)^n (\langle \pi \rangle - 1)^2 ab \\ &= (\langle \pi \rangle - 1)(a + b + (-2)^{n+1} ab). \end{aligned}$$

This now implies both claims.  $\square$

**Remark 4.8.** *The above can be reformulated in the language of contractions as follows: for  $n \geq 2$ , we have an isomorphism of abelian groups*

$$\mathbf{GW}_{-n}^\times(L) \cong W(L)_{\text{tor}}^{(n)}.$$

## 5. GERSTEN RESOLUTION AND STRICT $\mathbb{A}^1$ -INVARIANCE

In this section, we discuss a Gersten-type resolution for the unramified sheaf  $\mathbf{GW}^\times$ . The description of units in Proposition 2.2 provides an exact sequence which decomposes  $\mathbf{GW}^\times$  into a part coming from  $\mathbf{K}_1/2$  and a part  $\mathbf{NQ}$  which is closely related to the torsion in the fundamental ideal  $\mathbf{I}_{\text{tor}}$ . From the Gersten resolution, we can deduce strict  $\mathbb{A}^1$ -invariance.

**5.1. The unramified sheaf  $\mathbf{NQ}$ .** We first introduce an unramified sheaf  $NQ$ . Recall from Proposition 2.2 that the units in the Grothendieck-Witt ring decompose into a square-class part and a part coming from nilpotent elements in the Witt ring. The sheaf  $NQ$  deals with the part of the units of the Witt ring which comes from nilpotent elements:

**Definition 5.1.** *For a field  $F$ , we denote*

$$NQ(F) = (1 + I(F)_{\text{tor}}) / (F^\times / (F^\times)^2 \cap (1 + I(F)_{\text{tor}})) \cong W^\times(F) / (F^\times),$$

which is the quotient of the horizontal maps of the pushout square in Proposition 2.2.

The notation  $NQ$  is supposed to suggest that this group is the **Q**uotient of the subgroup of the units coming from **N**ilpotent elements of the Witt ring.

Since  $NQ$  is defined as the quotient of  $W^\times$  modulo the square classes  $\mathbb{G}_m/2$ , the data (D1)-(D3) for  $W^\times$  induce corresponding data for  $NQ$ . With these data,  $NQ$  extends to an unramified sheaf of abelian groups. For an irreducible smooth scheme  $X$  with function field  $F$ , we denote by

$$\mathbf{NQ}(X) = \bigcap_{x \in X^{(1)}} \mathbf{NQ}(\mathcal{O}_x) \subseteq \mathbf{NQ}(F)$$

the subgroup of unramified elements in  $NQ(F)$ .

**5.2. Gersten resolution for  $I_{\text{tor}}$ .** We next discuss the Gersten resolution for the torsion in the fundamental ideal of the Witt ring. The assignment  $L \in \mathcal{F}_k \mapsto I^*(L)$  mapping a field to the  $\mathbb{Z}$ -graded abelian group of powers  $I^n(L)$  of the fundamental ideal in the Witt ring  $W(L)$  extends to a  $\mathbb{Z}$ -graded family of strictly  $\mathbb{A}^1$ -invariant sheaves of groups  $\mathbf{I}^*$  on  $\text{Sm}_k$ . This is a consequence of Example 3.34, Lemma 3.35 and Theorem 2.46 of [Mor12].

Now we consider for  $n \in \mathbb{N}$  the morphism  $2^n : \mathbf{I}^* \rightarrow \mathbf{I}^*$ . As this concerns only the abelian group structure of  $I^*$ , it commutes with all the data (D1)-(D4) making  $\mathbf{I}^*$  an unramified sheaf of abelian groups. Using Lemma 3.35 and Theorem 2.46 of [Mor12] again, we have the following result:

**Lemma 5.2.** *The assignment*

$$I^*[2^n] : \mathcal{F}_k \rightarrow \mathcal{A}b : L \mapsto \ker(2^n : I^* \rightarrow I^*)$$

*extends to a strictly  $\mathbb{A}^1$ -invariant sheaf of abelian groups. The same is true for the filtered colimit*

$$I_{\text{tor}}^* = I^*[2^\infty] = \text{colim } I^*[2^n].$$

Using this together with the results from [Mor12], we have a Gersten resolution for  $\mathbf{I}_{\text{tor}}$ .

**Lemma 5.3.** (i) *There is a complex*

$$0 \rightarrow \mathbf{I}_{\text{tor}}(X) \rightarrow I(K)_{\text{tor}} \rightarrow \bigoplus_{x_1 \in X^{(1)}} W(\kappa(x_1))_{\text{tor}} \rightarrow \cdots \rightarrow \bigoplus_{x_n \in X^{(n)}} W(\kappa(x_n))_{\text{tor}} \rightarrow 0$$

*for any essentially smooth  $k$ -scheme  $X$  of dimension  $n$  with function field  $K$ .*

(ii) *This complex is a Nisnevich flasque resolution of  $\mathbf{I}_{\text{tor}}$ .*

(iii) *For  $X$  essentially smooth and local, the complex is exact.*

(iv) *The cohomology of this complex is  $\mathbb{A}^1$ -invariant for  $X$  an essentially smooth scheme.*

*Proof.* By [Mor12, Corollary 5.44], the Gersten resolution can be identified with what Morel calls Rost-Schmid complex.

(i) follows from [Mor12, Theorem 5.31], (ii) is [Mor12, Corollary 5.43], (iii) is [Mor12, Theorem 5.41], and (iv) is [Mor12, Theorem 5.38].  $\square$

**5.3. Resolution for  $\mathbf{NQ}$ .** The next step is an appropriate modification of the resolution for  $\mathbf{I}_{\text{tor}}$  to obtain a resolution for  $\mathbf{NQ}$ .

**Lemma 5.4.** (i) *There is a complex*

$$\begin{aligned} 1 \rightarrow \mathbf{1} + \mathbf{I}_{\text{tor}}(X) \rightarrow 1 + I(K)_{\text{tor}} &\rightarrow \bigoplus_{x_1 \in X^{(1)}} W(\kappa(x_1))_{\text{tor}}^{(1)} \rightarrow \\ &\rightarrow \cdots \rightarrow \bigoplus_{x_n \in X^{(n)}} W(\kappa(x_n))_{\text{tor}}^{(n)} \rightarrow 0 \end{aligned}$$

*for any essentially smooth  $k$ -scheme  $X$  of dimension  $n$  with function field  $K$ . The cohomology of this complex is  $\mathbb{A}^1$ -invariant for  $X$  an essentially smooth scheme, and the complex is exact for  $X$  an essentially smooth local scheme.*

(ii) *The complex of (i) contains the following subcomplex:*

$$1 \rightarrow \mathbb{G}_m/2(X) \cap \mathbf{1} + \mathbf{I}_{\text{tor}}(X) \rightarrow (K^\times/(K^\times)^2) \cap (1 + I(K)_{\text{tor}}) \rightarrow \bigoplus_{x_1 \in X^{(1)}} \mathcal{S}_{\pi(x_1)} \rightarrow 0,$$

*where  $\pi(x_1)$  is a uniformizer for the point  $x_1$  and  $\mathcal{S}_{\pi(x_1)} = \mathbb{Z}/2$  or 0 if the square class of the uniformizer  $\pi$  is a sum of squares or not, respectively.*

The cohomology of this complex is  $\mathbb{A}^1$ -invariant for  $X$  an essentially smooth scheme, and the complex is exact for  $X$  an essentially smooth local scheme.

(iii) The statements in (i) also hold for

$$1 \rightarrow \mathbf{NQ}(X) \rightarrow NQ(K) \rightarrow \bigoplus_{x_1 \in X^{(1)}} W(\kappa(x_1))_{\text{tor}}^{(1)} / \mathcal{S}_{\pi(x_1)} \rightarrow$$

$$\rightarrow \cdots \rightarrow \bigoplus_{x_n \in X^{(n)}} W(\kappa(x_n))_{\text{tor}}^{(n)} \rightarrow 0$$

*Proof.* (i) is a direct consequence of Lemma 5.3. The complex is obtained by taking the complex from Lemma 5.3, and changing the group structures:  $1 + I(F)_{\text{tor}}$  as a set is the same as  $I(F)_{\text{tor}}$ , but with multiplication instead of addition. Similarly, the underlying sets of  $W(F)_{\text{tor}}^{(n)}$  and  $W(F)_{\text{tor}}$  are the same, but the addition is different. From Section 4, we know that the differentials of the complex of Lemma 5.3 are homomorphisms for the modified group structures. Moreover, the neutral elements for  $W(F)_{\text{tor}}^{(n)}$  and  $W(F)_{\text{tor}}$  are the same. Therefore, the property  $\partial^2 = 0$  claimed in the present lemma does not depend on the abelian group structure, only on the underlying set maps and the neutral elements - hence it follows from Lemma 5.3. Similarly, vanishing of cohomology in the case of smooth local schemes does not depend on the abelian group structure, only on the underlying sets and set maps. Hence it follows from Lemma 5.3.

(ii) From the Gersten resolution for  $K_1^M/2$ , we know that the claim holds for

$$1 \rightarrow \mathbb{G}_m/2(X) \rightarrow K^\times / (K^\times)^2 \rightarrow \bigoplus_{x_1 \in X^{(1)}} \mathbb{Z}/2 \rightarrow 0.$$

Now any  $u \in K^\times$ , is either unramified or generates the corresponding  $\mathbb{Z}/2$ -copy in the quotient. Therefore, restricting to the square classes in  $1 + I(K)_{\text{tor}}$  provides the diagram in (ii), hence proves the claim. In particular, the complex is exact for  $X$  an essentially smooth local scheme.

(iii) follows by taking the quotient of the complex in (i) by the subcomplex in (ii). Exactness for essentially smooth local schemes follows by the long exact sequence for the corresponding exact sequence of complexes. The same is true for  $\mathbb{A}^1$ -invariance.  $\square$

#### 5.4. Resolution for $\mathbf{GW}^\times$ .

**Proposition 5.5.** *Let  $X$  be an essentially smooth scheme of dimension  $n$  over  $X$  with function field  $K$ . We have the following exact sequence of complexes:*

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathbb{G}_m/2(X) & \longrightarrow & \mathbf{GW}^\times(X) & \longrightarrow & \mathbf{NQ}(X) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & K^\times/(K^\times)^2 & \longrightarrow & GW(K)^\times & \longrightarrow & NQ(K) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \bigoplus \mathbb{Z}/2 & \longrightarrow & \bigoplus \left( \mathcal{A}_{x_1} \oplus W(\kappa(x_1))_{\text{tor}}^{(1)} \right) & \longrightarrow & \bigoplus W(\kappa(x_1))_{\text{tor}}^{(1)} / \mathcal{S}_{x_1} & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & \longrightarrow & \bigoplus W(\kappa(x_2))_{\text{tor}}^{(2)} & \longrightarrow & \bigoplus W(\kappa(x_2))_{\text{tor}}^{(2)} & \longrightarrow 0 \\
& & & \downarrow & & \downarrow & \\
& & & \vdots & & \vdots & \\
& & & \downarrow & & \downarrow & \\
0 \longrightarrow & \bigoplus W(\kappa(x_n))_{\text{tor}}^{(n)} & \longrightarrow & \bigoplus W(\kappa(x_n))_{\text{tor}}^{(n)} & \longrightarrow & 0 & \\
& & & \downarrow & & \downarrow & \\
& & & 0 & & 0 & 
\end{array}$$

The vertical morphisms are the boundary maps discussed in Section 4. The omitted index sets of the direct sums are the respective sets of codimension  $i$  points  $x_i \in X^{(i)}$ .

*Proof.* (i) The diagram is commutative. This is almost by definition: the second line is the one coming from the definition of  $NQ$  resp. the pushout square of Proposition 2.2. The first line is the restriction to unramified elements, so these squares commute. The further lines are those computed in Section 4, so these squares also do commute.

(ii) The left column is a complex. In fact, it is the Gersten resolution for the strictly  $\mathbb{A}^1$ -invariant sheaf  $\mathbf{K}_1^M/2$ .

(iii) The right column is a complex by Lemma 5.4.

(iv) The rows of the diagram are exact. This follows from the description of units, cf. Proposition 2.2. In fact,  $NQ$  is defined in such a way that the second row is exact, whence also the exactness of the first row. In the third row, we have the following case distinction: if  $\pi$  is a sum of squares, then  $\mathcal{A} = 0$  and  $\mathcal{S} \cong \mathbb{Z}/2$ , in which case we have an exact sequence dividing out a copy of  $\mathbb{Z}/2$  from  $W(\kappa(x_1))_{\text{tor}}^{(1)}$ . If  $\pi$  is not a sum of squares, then  $\mathcal{A} \cong \mathbb{Z}/2$ , and  $\mathcal{S} = 0$ , in which case we have an exact sequence dividing out the additional  $\mathbb{Z}/2$  in the middle. All rows below the third are obviously exact.

(v) It follows by a diagram chase that the middle column is also a complex.  $\square$

**5.5. Strict  $\mathbb{A}^1$ -invariance.** Recall that a sheaf of groups  $\mathcal{G}$  on  $\text{Sm}_k$  is called *strictly  $\mathbb{A}^1$ -invariant* if for each  $i$  and each smooth scheme  $X$  over  $k$  the associated morphism  $H_{\text{Nis}}^i(X, \mathcal{G}) \rightarrow H_{\text{Nis}}^i(X \times \mathbb{A}^1, \mathcal{G})$  is an isomorphism.

**Theorem 5.6.** *Let  $X$  be an essentially smooth local scheme of dimension  $n$  over  $k$ , let  $K$  denote its function field, and let  $z$  denote its closed point. Then the following sequence is exact:*

$$\begin{aligned}
0 \rightarrow \mathbf{GW}^\times(X) &\rightarrow GW(K)^\times \rightarrow \bigoplus_{x_1 \in X^{(1)}} \left( \mathcal{A}_x \oplus W(\kappa(x_1))_{\text{tor}}^{(1)} \right) \rightarrow \\
&\rightarrow \bigoplus_{x_2 \in X^{(2)}} W(\kappa(x_2))_{\text{tor}}^{(2)} \rightarrow \cdots \rightarrow \bigoplus_{x_{n-1} \in X^{(n-1)}} W(\kappa(x_{n-1}))_{\text{tor}}^{(n-1)} \rightarrow W(\kappa(z))_{\text{tor}}^{(n)} \rightarrow 0.
\end{aligned}$$

*Proof.* Using Proposition 5.5, we immediately reduce to the exactness of the resolution of  $\mathbf{NQ}$  which follows from Lemma 5.4.  $\square$

**Theorem 5.7.** *The unramified sheaf  $\mathbf{GW}^\times$  is strictly  $\mathbb{A}^1$ -invariant.*

*Proof.* Again, we use Proposition 5.5 and the Gersten resolution for  $K_1^M/2$  to reduce to the  $\mathbb{A}^1$ -invariance of  $\mathbf{NQ}$  which follows Lemma 5.4.  $\square$

From the description of the units in the Grothendieck-Witt ring, we obtain a long exact sequence decomposing the  $\mathbf{GW}^\times$ -homology into a weight one  $K$ -theory part and a modified Witt-ring homology part. This is a direct consequence of Proposition 5.5.

**Proposition 5.8.** *There is a short exact sequence of strictly  $\mathbb{A}^1$ -invariant sheaves of groups on  $\mathrm{Sm}_k$*

$$1 \rightarrow \mathbb{G}_m/2 \rightarrow \mathbf{GW}^\times \rightarrow \mathbf{NQ} \rightarrow 1.$$

*This induces (functorially on  $\mathrm{Sm}_k$ ) an exact sequence*

$$\begin{aligned} 0 \rightarrow \mathbb{G}_m/2(X) \rightarrow \mathbf{GW}^\times(X) \rightarrow \mathbf{NQ}(X) \rightarrow \mathrm{Pic}/2(X) \rightarrow \\ \rightarrow H_{\mathrm{Nis}}^1(X, \mathbf{GW}^\times) \rightarrow H_{\mathrm{Nis}}^1(X, \mathbf{NQ}) \rightarrow 0 \end{aligned}$$

*as well as (for  $i \geq 2$ ) isomorphisms*

$$H_{\mathrm{Nis}}^i(X, \mathbf{GW}^\times) \xrightarrow{\cong} H_{\mathrm{Nis}}^i(X, \mathbf{NQ}).$$

Note that from the proof above, we have a bijection of sets  $H^i(X, \mathbf{NQ}) \cong H^i(X, \mathbf{W}_{\mathrm{tor}})$  for  $i \geq 2$ . However, as the abelian group structures of  $W_{\mathrm{tor}}$  and  $W_{\mathrm{tor}}^{(n)}$  are different, the above bijection of sets does not respect the group structure. Nevertheless, non-triviality of elements in  $H^i(X, \mathbf{NQ})$  can be detected in  $H^i(X, \mathbf{W}_{\mathrm{tor}})$ .

## 6. $\mathbb{A}^1$ -SPHERICAL FIBRATIONS AND ORIENTATION THEORY

In this final section, we discuss consequences of the previous results for orientation theory and spherical fibrations in  $\mathbb{A}^1$ -homotopy theory.

### 6.1. Classifying space of spherical fibrations.

**Proposition 6.1.** *The space  $B\mathrm{hAut}_\bullet(S^{2n,n})$  is  $\mathbb{A}^1$ -local. Hence it is in fact the classifying space of (Nisnevich locally trivial) spherical fibrations.*

*Proof.* This is a direct consequence of Theorem 5.7 and [Wen13, Theorem 8.1] resp. [Wen13, Corollary 8.2].  $\square$

As a consequence, we can consider an unstable version of the  $J$ -homomorphism on the classifying space level.

**Proposition 6.2.** *The homomorphism  $GL_n \rightarrow \mathrm{hAut}_\bullet S^{2n,n}$  induces a morphism of  $\mathbb{A}^1$ -local simplicial sheaves*

$$J^{\mathbb{A}^1}(n) : L_{\mathbb{A}^1} BGL_n \rightarrow B\mathrm{hAut}_\bullet(S^{2n,n}).$$

*Proof.* We describe the homomorphism: a matrix  $M \in GL_n$  induces a scheme morphism  $M : \mathbb{A}^n \rightarrow \mathbb{A}^n$ . Since this morphism preserves  $\mathbb{A}^n \setminus \{0\}$ , it descends to a morphism

$$M : \mathbb{A}^n/(\mathbb{A}^n \setminus \{0\}) \rightarrow \mathbb{A}^n/(\mathbb{A}^n \setminus \{0\}).$$

This construction evidently maps matrix multiplication to composition. We choose a functorial fibrant replacement  $\mathbb{A}^n/(\mathbb{A}^n \setminus \{0\}) \rightarrow S^{2n,n}$  (note the abuse of notation), and have thus described the homomorphism  $GL_n \rightarrow \mathrm{hAut}_\bullet S^{2n,n}$ . A homomorphism between simplicial monoids induces a morphism of the corresponding

simplicial classifying spaces, and we get the required morphism between the  $\mathbb{A}^1$ -local simplicial sheaves by a functorial fibrant replacement again.  $\square$

The morphisms  $BGL_n \rightarrow B\mathrm{hAut}_\bullet(S^{2n,n})$  stabilize to a morphism  $BGL_\infty \rightarrow B\mathrm{hAut}_\bullet(S^{2\infty,\infty})$ , where on the source we stabilize by adding a trivial line bundle, and on the target we stabilize by fibrewise suspension with  $S^{2,1}$ . We denote by  $G/O(n)$  the space

$$G/O(n) = \mathrm{hofib}\left(J^{\mathbb{A}^1}(n) : L_{\mathbb{A}^1}BGL_n \rightarrow B\mathrm{hAut}_\bullet(S^{2n,n})\right), n \in \mathbb{N} \cup \{\infty\}.$$

This space controls the difference between the classification of rank  $n$  vector bundles and the rank  $n$  classification of spherical fibrations. The notation is chosen to fit the usual topological notation in which  $G/O$  is the homotopy fibre of the  $J$ -homomorphism  $BO \rightarrow BG$ .

It would be very interesting to study the morphism  $[X, J^{\mathbb{A}^1}(n)] : [X, BGL_n]_{\mathbb{A}^1} \rightarrow [X, B\mathrm{hAut}_\bullet(S^{2n,n})]_{\mathbb{A}^1}$ , i.e. really compare vector bundles and spherical fibrations. However, lack of descriptions of higher homotopy groups of  $B\mathrm{hAut}_\bullet(S^{2n,n})$  restricts us to the study of the first Postnikov section of  $J^{\mathbb{A}^1}(n)$ , i.e. the homomorphism

$$[X, J^{\mathbb{A}^1}(n)^{(1)}] : [X, B\mathbb{G}_m]_{\mathbb{A}^1} \rightarrow [X, B\pi_1^{\mathbb{A}^1} B\mathrm{hAut}_\bullet(S^{2n,n})]_{\mathbb{A}^1} \cong [X, B\mathbf{GW}^\times]_{\mathbb{A}^1}.$$

This is what we will do in the rest of the section.

**6.2. Cohomology with unit coefficients.** In this section, we discuss the cohomology of  $\mathbf{GW}^\times$ .

**Proposition 6.3.** *We have*

$$\pi_1^{\mathbb{A}^1} J^{\mathbb{A}^1}(n) : \mathbb{G}_m \rightarrow \mathbf{GW}^\times : u \in F^\times \mapsto \langle u \rangle \in GW(F)^\times.$$

The composition  $\mathrm{Pic}(X) \rightarrow \mathrm{Pic}/2(X) \rightarrow H_{\mathrm{Nis}}^1(X, \mathbf{GW}^\times)$  of the obvious projection with the morphism from Proposition 5.8 is precisely the induced morphism

$$H_{\mathrm{Nis}}^1(X, J^{\mathbb{A}^1}(n)^{(1)}) : H_{\mathrm{Nis}}^1(X, \mathbb{G}_m) \rightarrow H_{\mathrm{Nis}}^1(X, \mathbf{GW}^\times).$$

*Proof.* From the description of  $J$ , to determine the morphism induced on fundamental groups, we need to consider for  $u \in k^\times$  the homotopy class of multiplication with the matrix  $\mathrm{diag}(u, 1, \dots, 1)$  in  $[\mathbb{A}^n \setminus \{0\}, \mathbb{A}^n \setminus \{0\}]_{\mathbb{A}^1}$ . Under the isomorphism  $[\mathbb{A}^n \setminus \{0\}, \mathbb{A}^n \setminus \{0\}]_{\mathbb{A}^1} \cong GW(k)$ , this is mapped to  $\langle u \rangle$ , cf. [Mor12].  $\square$

**Remark 6.4.** *Note that the above implies that the image of  $\mathbb{G}_m \rightarrow \mathbf{GW}^\times$  is then determined by the image of  $\mathbb{G}_m \rightarrow \mathbf{W}^\times$  since it factors through the  $W^\times$ -coset of 1 in the extension*

$$1 \rightarrow \{\pm 1\} \rightarrow \mathbf{GW}^\times \rightarrow \mathbf{W}^\times \rightarrow 1.$$

**Proposition 6.5.** *We have the following computation of Nisnevich cohomology of  $\mathbf{GW}^\times$  over spheres:*

$$H_{\mathrm{Nis}}^i(S^p \wedge \mathbb{G}_m^{\wedge q}, \mathbf{GW}^\times) \cong H_{\mathrm{Nis}}^{i-p}(\mathrm{Spec} k, (\mathbf{GW}^\times)_{-q}) \cong \begin{cases} 0 & i \neq p \\ GW(k)^\times & q = 0, \\ \mathcal{A} \oplus W(k)_{\mathrm{tor}}^{(1)} & q = 1, \\ W(k)_{\mathrm{tor}}^{(q)} & q \geq 2. \end{cases}$$

In the above,  $\mathcal{A} = 0$  if  $T \in k(T)$  is a sum of squares and  $\mathcal{A} \cong \mathbb{Z}/2$  otherwise.

*Proof.* The first isomorphism is standard, it could be called “Thom isomorphism”. The second restates the computations from Section 4.  $\square$



**Remark 6.6.** The tangent bundle of  $\mathbb{P}^1$  is  $\mathcal{O}(-2)$ , hence its class in the cohomology group  $H_{\text{Nis}}^1(\mathbb{P}^1, \mathbf{GW}^\times)$  is trivial. The  $\mathbb{Z}/2$ -summand in the cohomology group is generated by  $\mathcal{O}(-1)$ , the Möbius strip.

There can be many additional  $\mathbb{A}^1$ -homotopy classes of morphisms  $\mathbb{P}^1 \rightarrow B\mathbf{GW}^\times$ . For example, if  $k = \mathbb{Q}$ , the Witt ring  $W(k)$  contains a lot of torsion elements. However, none of these homotopy classes is visible in either real or complex realization: for the complex realization  $W(\mathbb{C}) = \mathbb{Z}/2$ , for the real realization  $W(\mathbb{R}) = \mathbb{Z}$  is torsion-free.

### 6.3. Orientation theory.

**Definition 6.7.** A spherical fibration  $f \in [X, B\text{hAut}_\bullet(S^{2n,n})]_{\mathbb{A}^1}$  is called orientable if it factors through the universal  $\mathbb{A}^1$ -covering

$$B\widetilde{\text{hAut}_\bullet(S^{2n,n})} = B(\text{hAut}_\bullet(S^{2n,n})_0) \rightarrow B\text{hAut}_\bullet(S^{2n,n}).$$

Given a spherical fibration  $f : X \rightarrow B\text{hAut}_\bullet(S^{2n,n})$  with  $X$  an  $\mathbb{A}^1$ -connected space, we define its orientation character to be the morphism

$$\pi_1^{\mathbb{A}^1}(f) : \pi_1^{\mathbb{A}^1}(X) \rightarrow \pi_1^{\mathbb{A}^1} B\text{hAut}_\bullet(S^{2n,n}) \cong \mathbf{GW}^\times.$$

**Remark 6.8.** • Obviously, a spherical fibration over an  $\mathbb{A}^1$ -simply-connected base  $X$  is orientable.

- There is a similar notion of orientability for vector bundles requiring that the classifying map  $X \rightarrow BGL_n$  factors through the inclusion  $BSL_n \rightarrow BGL_n$ . This is equivalent to the vector bundle having trivial first Chern class.
- More generally, we could define an orientation character to be the induced morphism on fundamental groupoids in case the space  $X$  is not  $\mathbb{A}^1$ -connected. As we only deal with  $\mathbb{P}^1$  in what follows, we keep it simple.

The above notion of orientability is the one employed in topology for spherical fibrations. In topology, orientability of vector bundles and their associated spherical fibrations are equivalent since  $\pi_1(O(n)) \cong \pi_1(G) \cong \mathbb{Z}/2$ . However, this is not the case in  $\mathbb{A}^1$ -homotopy theory. The following comparison of the two notions of orientability is a direct consequence of Proposition 5.8:

**Proposition 6.9.** Let  $f \in [X, BGL_n]_{\mathbb{A}^1}$  be a (continuous) vector bundle on  $X$ . The associated spherical fibration is orientable if one of the following conditions holds:

- (i)  $c_1(E) \equiv 0 \pmod{2}$ .
- (ii) The class  $[f] \in \text{Pic}/2$  is in the image of  $\mathbf{NQ}$  of Proposition 5.8.

We use the proposition to discuss some examples of the arithmetic nature of the above notion of orientability.

**Example 6.10.** The orientation character of the tangent bundle of  $\mathbb{P}^n$  is given by  $\mathbb{G}_m \rightarrow \mathbf{GW}^\times : T \mapsto \langle T^{n+1} \rangle$ . We denote by  $ST(n)$  the spherical fibration associated to the tangent bundle of  $\mathbb{P}^n$ . We use that we can determine the morphism  $\mathbb{G}_m \rightarrow \mathbf{GW}^\times$  from its image in  $\mathbf{W}^\times$ , cf. Remark 6.4.

- (i) Over an algebraically closed base  $k = \bar{k}$ ,  $ST(n)$  is orientable for any  $n$ : in that case  $W(k) = \mathbb{Z}/2$ , so  $W(k)^\times = \{1\}$ .
- (ii) Over the real numbers  $k = \mathbb{R}$ , the morphism  $\pi_1^{\mathbb{A}^1} BGL_n \rightarrow \mathbf{W}^\times$  looks as follows:

$$k^\times \rightarrow W(\mathbb{R})^\times : u \mapsto \begin{cases} 1 & u \in (\mathbb{R}^\times)^2 \\ -1 & u \notin (\mathbb{R}^\times)^2 \end{cases}$$

Composing this with the orientation character we find that  $ST(n)$  is orientable if and only if  $n$  is odd. This is the behaviour of orientability for real projective spaces.

(iii) In characteristic  $p$ , every element is a sum of squares. Therefore,  $\langle T \rangle$  lies in the image of  $\mathbf{NQ}$ . Hence, in characteristic  $p$ ,  $ST(n)$  is always orientable.  $\square$

**6.4. Reducing spherical fibrations to vector bundles.** We comment on the first obstruction for spherical fibrations to be the induced spherical fibrations of vector bundles.

**Proposition 6.11.** *We have an isomorphism of strongly  $\mathbb{A}^1$ -invariant sheaves of groups*

$$\pi_0^{\mathbb{A}^1}(G/O) \cong \mathbf{NQ}.$$

*In particular, the space  $G/O$  is étale  $\mathbb{A}^1$ -connected but not necessarily  $\mathbb{A}^1$ -connected.*

*Let  $x \in G/O(\mathrm{Spec} k)$  be in the component of the trivial spherical fibration. Then there is a surjection*

$$\pi_1^{\mathbb{A}^1}(G/O, x) \rightarrow 2\mathbb{G}_m \rightarrow 0.$$

*Proof.* The part concerning  $\pi_0^{\mathbb{A}^1}$  and  $\pi_1^{\mathbb{A}^1}$  of the long exact sequence associated to the fibre sequence

$$G/O \rightarrow BGL_n \rightarrow B\mathrm{hAut}_\bullet(S^{2n,n})$$

is exactly the exact sequence of Proposition 5.8. This implies the result on  $\pi_0$ , noting that both  $BGL_n$  and  $B\mathrm{hAut}_\bullet(S^{2n,n})$  are  $\mathbb{A}^1$ -connected.

The result on  $\pi_1$  also follows from the long exact sequence, noting that  $\mathbb{G}_m \cong \pi_1^{\mathbb{A}^1} BGL_n \rightarrow \mathbf{GW}^\times$  factors through the quotient  $\mathbb{G}_m/2$ .  $\square$

**Proposition 6.12.** *Let  $C$  be a smooth curve. Then we have*

$$[C, B\mathrm{hAut}_\bullet(S^{2n,n})]_{\mathbb{A}^1} \cong [C, B\mathbf{GW}^\times]_{\mathbb{A}^1} \cong H_{\mathrm{Nis}}^1(C, \mathbf{GW}^\times).$$

*Proof.* The last bijection is basically the definition of  $B\mathbf{GW}^\times$ .

The first bijection is obstruction theory: the space  $B\mathbf{GW}^\times$  is the first Postnikov section of  $B\mathrm{hAut}_\bullet(S^{2n,n})$ . The obstruction and lifting classes for lifting a morphism  $f : C \rightarrow B\mathrm{hAut}_\bullet(S^{2n,n})^{(i)}$  to the next stage of the Postnikov tower lie in  $H_{\mathrm{Nis}}^{i+1}$  resp.  $H_{\mathrm{Nis}}^{i+2}$  of  $C$  with coefficients in  $\pi_{i+1}^{\mathbb{A}^1} B\mathrm{hAut}_\bullet(S^{2n,n})$ . But these cohomology groups vanish for reasons of cohomological dimension. Therefore, any morphism  $f \in [C, B\mathbf{GW}^\times]_{\mathbb{A}^1}$  has a (up to  $\mathbb{A}^1$ -homotopy) unique extension to  $B\mathrm{hAut}_\bullet(S^{2n,n})$ .  $\square$

**Corollary 6.13.** *We have*

$$[\mathbb{P}^1, B\mathrm{hAut}_\bullet(S^{2n,n})]_{\mathbb{A}^1} \cong \mathcal{S} \oplus W(k)_{\mathrm{tor}}^{(1)}.$$

*Proof.* This follows directly from Proposition 6.12 and Proposition 6.5.  $\square$

Therefore, we can explicitly describe all spherical fibrations over  $\mathbb{P}^1$ . They are given by glueing two trivial spherical fibrations over  $\mathbb{P}^1 \setminus \{0\}$  and  $\mathbb{P}^1 \setminus \{\infty\}$  along a transition morphism  $\mathbb{G}_m \rightarrow \mathrm{hAut}_\bullet S^{2n,n}$ .

In particular, there exist spherical fibrations over  $\mathbb{P}^1$  that are not associated to vector bundles. By Proposition 5.8, these are classified exactly by

$$H_{\mathrm{Nis}}^1(\mathbb{P}^1, \mathbf{NQ}) \cong H_{\mathrm{Nis}}^0(\mathrm{Spec} k, (\mathbf{NQ})_{-1}) \cong W(k)_{\mathrm{tor}}^{(1)}/\mathcal{S}.$$

**Example 6.14.** *Cazanave has described the automorphisms of  $\mathbb{P}^1$  in [Caz08]. As a consequence of the results of [Caz08], the rational function*

$$\frac{X^3 - \left(\frac{a_3}{a_2} + \frac{a_2}{a_1}\right)X}{a_1 X^2 - \frac{a_1 a_3}{a_2}}$$

is the endomorphism associated to the quadratic form  $\langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \langle a_3 \rangle$  in  $GW(k)$ .

In particular, for  $k = \mathbb{Q}_p$  and  $u$  any square class, the rational function

$$\frac{X^3 - (T + u)X}{X^2 - T}$$

has associated quadratic form  $1 \oplus \langle T \rangle \oplus \langle u \rangle$ . Therefore, the above rational function is an endomorphism of  $\mathbb{P}^1$  which is invertible up to  $\mathbb{A}^1$ -homotopy, but not  $\mathbb{A}^1$ -homotopic to a fractional linear transformation. Tracing through our identifications Proposition 6.5 and Lemma 4.3, we find that gluing two trivial spherical fibrations over  $\mathbb{P}^1 \setminus \{0\}$  and  $\mathbb{P}^1 \setminus \{\infty\}$  along the  $\mathbb{G}_m$ -family of endomorphisms given by the rational functions

$$\frac{X^3 - (T + u)X}{X^2 - T}$$

provides a spherical fibration over  $\mathbb{P}^1$  whose associated class in  $H_{\text{Nis}}^1(\mathbb{P}^1, \mathbf{GW}^\times)$  is non-trivial. Similar examples can be given using quadratic forms over  $\mathbb{Q}$ .  $\square$

The spherical fibrations above provide examples of exotic Poincaré duality structure on  $\mathbb{P}^1$ . In the case of the base field  $\mathbb{Q}$ , these structures are indistinguishable from the standard Poincaré duality on  $\mathbb{P}^1$  in real and complex realizations, in the category of motives and in the  $\mathbb{Z}[1/2]$ -localized stable  $\mathbb{A}^1$ -homotopy. In particular, there are several spherical fibrations inducing the standard Poincaré duality, but only one of them is induced from a vector bundle. The notion of Poincaré duality space, should it ever take form in  $\mathbb{A}^1$ -homotopy, necessarily has to include the 2-torsion information from the Witt group to prevent this sort of pathological behaviour.

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MATTHIAS WENDT, MATHEMATISCHES INSTITUT, ALBERT-LUDWIGS-UNIVERSITÄT FREIBURG, ECKERSTRASSE 1, 79104, FREIBURG IM BREISGAU, GERMANY

E-mail address: matthias.wendt@math.uni-freiburg.de